Nine-moment phonon hydrodynamics based on the modified Grad-type approach: hyperbolicity of the one-dimensional flow

2004 J. Phys. A: Math. Gen. 37 11053

(http://iopscience.iop.org/0305-4470/37/45/021)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **37** (2004) 11053–11072 PII: S0305-4470(04)80158-6

Nine-moment phonon hydrodynamics based on the modified Grad-type approach: hyperbolicity of the one-dimensional flow

Zbigniew Banach[1] **and Wieslaw Larecki**[2]

[1] Department of Fluid Mechanics, Centre of Mechanics, Institute of Fundamental Technological Research, Polish Academy of Sciences, Swietokrzyska 21, 00-049 Warsaw, Poland
[2] Department of Theory of Continuous Media, Institute of Fundamental Technological Research, Polish Academy of Sciences, Swietokrzyska 21, 00-049 Warsaw, Poland

E-mail: zbanach@ippt.gov.pl and wlarecki@ippt.gov.pl

Received 4 May 2004, in final form 24 September 2004
Published 28 October 2004
Online at stacks.iop.org/JPhysA/37/11053
doi:10.1088/0305-4470/37/45/021

Abstract

After expanding the distribution function about an anisotropic Planck function, the new moment closure method of Banach and Larecki applied to the Boltzmann–Peierls equation for the phonon gas dynamics leads to a whole hierarchy of closed systems of moment equations. The system of equations for the energy density and the heat flux is the first, non-perturbative member of this hierarchy of closures. In our previous paper (2004 *J. Phys. A: Math. Gen.* **37** 9805), emphasis was placed on deriving the next member, the 9-moment anisotropic closure that involves the flux of the heat flux as an extra gas-state variable. Here, as a first step in effectively analysing this system, we present a study of the one-dimensional, rotationally symmetric reduction of these equations. Under the assumption of Callaway's model, a systematic procedure is derived which shows that the obtained system of three evolution equations for three nonvanishing gas-state variables can be cast into a symmetric hyperbolic form. For the sake of completeness, we describe explicitly the region of symmetric hyperbolicity in parameter space (the space defined by the gas-state variables). The evolution system is symmetric hyperbolic for significant ranges of physical conditions, i.e., there are effectively no unphysical limitations on the magnitude of the energy density and the heat flux. This paper also deals with the eigenvalue problem and calculates approximately the characteristic speeds.

PACS numbers: 51.10.+y, 44.10.+i, 05.30.Jp, 05.70.Ln

1. Introduction

Accurate prediction of flows of non-equilibrium phonon gases requires a consideration of the Boltzmann–Peierls (BP) equation [1–3], which governs the time evolution of the distribution function describing the number density of phonons at position (x^i) having wave vector $\mathbf{k}$. The mathematical and computational difficulties associated with this equation are well known. It is therefore of interest to study approximating phonon dynamic systems which are related to moments of the BP equation. The simplest such system is the system of equations for the energy density and the heat flux. This system has four equations and is derived, e.g., in [4].

A systematic expansion of the number density of phonons about an anisotropic Planck function[3], following the Banach–Larecki procedure [5], leads to a whole hierarchy of closed systems of moment equations. The aforementioned system of four equations is the first, non-perturbative member of this hierarchy of closures. In [5], we have investigated in detail the next member, the 9-moment anisotropic closure. Denoting by $\mathcal{M}^{ij}$ the flux of the heat flux and by M^{ij} the deviatoric part of $\mathcal{M}^{ij}$, the 9-moment closure yields a system of nine transport equations with the independent variables being, in addition to the energy density ϵ and three components (q^i) of the heat flux $\mathbf{q}$, five components of a symmetric tracefree tensor M^{ij}. Our approach is fundamentally different from a more traditional approach [6] which uses the equilibrium Planck distribution as the base. The main advantage of using the anisotropic Planck function is that the heat flux is incorporated into the model in a non-perturbative manner, thereby allowing virtually arbitrarily large values for the components of this heat flux. Moreover, with this approach one can describe phenomena at frequencies comparable to the inverse of the normal time. We consider the case when the effective relaxation time $\tilde{\tau}_n$ for normal processes is much smaller than the effective relaxation time τ_r for resistive processes. During the first time period, the normal time, the number density of phonons relaxes to an anisotropic Planck function, and then during the longer, resistive time, this number density settles into an equilibrium Planck distribution. More details on these issues can be found in [5].

Even under the assumption of Callaway's model [7], the 9-moment anisotropic closure forms a rather complicated system of evolution equations. As a first step in effectively analysing this system, we present here a study of the one-dimensional, rotationally symmetric reduction of these equations. Since the obtained system of three equations contains only three gas-state variables, it is then possible to answer the question whether the field equations governing specific one-dimensional motions are hyperbolic. Clearly, our system will be hyperbolic in a convex set of gas-state variables if the eigenvalue problem has three real roots (not necessarily distinct) and if the corresponding eigenvectors span the space of three dimensions [8–10].

The difficulty associated with solving the eigenvalue problem lies in the complex nature of the characteristic polynomial, which is due to the fact that the heat flux is not a perturbative quantity. This difficulty may be overcome by means of an alternative technique, similar to that employed by Larecki [11, 12] who considered symmetrizable hyperbolic systems [13, 14]. Given this technique, the present work aims to proceed in two steps. Upon writing the evolution system in a normal Cauchy form, we first prove that this system admits a family of left symmetric symmetrizers characterized by three arbitrary functions of gas-state variables. The second step involves choosing these functions so that the resulting symmetrizer is positive definite. This is achieved through the use of an approximate expression for the Boltzmann entropy[4]. We are then able to conclude that the evolution equations are symmetrizable

[3] In [5], this function is also called the quasi-equilibrium Planck distribution.
[4] For the details concerning this approximate expression, see our discussion in [5, section 3.4].

hyperbolic equations [13]. For a physicist, this is especially attractive for at least three reasons. First, there do exist mathematical proofs that show (local) existence and uniqueness of solutions to the related Cauchy initial-value problem. Second, symmetric hyperbolic systems preclude action at a distance and ensure finite speeds. Finally, such systems allow for the propagation of data that are non-analytic across certain (characteristic) surfaces, which can be interpreted as wavefronts. A standard reference on these aspects is the book by Courant and Hilbert [15].

For the sake of completeness, we also specify the region of symmetric hyperbolicity in parameter space, the space defined by the gas-state variables ϵ, q^1 and M^{11}. Later, in section 3.5, we decide to introduce the expansion coefficient $\varphi^{0|3}$ in place of M^{11}. It is significant that there are effectively no unphysical limitations on the values of the energy density ϵ and the magnitude of the nonvanishing component q^1 of the heat flux $\mathbf{q}$. However, since we were forced to make approximations [5] in order to obtain tractable equations, it may be suspected that the evolution system is not symmetric hyperbolic for all ranges of formally possible conditions[5]. In fact, we shall find this suspicion confirmed and shall calculate the region of symmetric hyperbolicity explicitly. At the same time, it seems important to note that the transport equations for (ϵ, q^1, M^{11}) or $(\epsilon, q^1, \varphi^{0|3})$ form a symmetrizable hyperbolic system even beyond the limits of their original derivation ($|\varphi^{0|3}| \ll 1$), and indeed this type of observation is one of the most unexpected features of our approach.

We finally mention the following. A non-perturbative method to derive a hierarchy of closed systems of moment equations is to use the closure by entropy maximization [16–19]. Unlike the perturbative approaches [20–22], the basic advantage of using this method is that if one expresses the moment densities and the collision productions in terms of Lagrange multipliers, the evolution equations for these multipliers are then automatically symmetric hyperbolic at every order of truncation. However, the difficulty is that various moment fluxes and collisional terms are impossible to evaluate explicitly as functions of the gas-state variables. Moreover, cases are known where the method of maximum entropy is ill-defined in a neighbourhood of equilibrium (quasi-equilibrium) states [18, 23]. This is not a problem here, but the former difficulty constitutes the biggest obstacle to any practical implementation of this non-perturbative method.

Our paper is organized as follows. Section 2 is devoted to the study of the one-dimensional, rotationally symmetric reduction of 9-moment phonon hydrodynamics. The purpose of section 3 is to transform the evolution system into a quasi-linear, symmetric hyperbolic form. In section 4, we deal briefly with the eigenvalue problem and calculate approximately the characteristic speeds by expanding certain nonlinear functions of the gas-state variables as polynomials with respect to the nonvanishing component q^1 of the heat flux. Section 5 is for discussion and final remarks. Some intermediate calculations are put into the appendix.

2. One-dimensional motions

2.1. Reduction to three gas-state variables

In [5], the equations of 9-moment phonon hydrodynamics were derived using the modified Grad-type approach. These equations may be written in terms of the energy density ϵ, the heat flux q^i and the deviatoric part M^{ij} of the flux of the heat flux $\mathcal{M}^{ij}$, as

[5] By this we mean the following. In [5], it was assumed that the expansion coefficients $\varphi^{0|3}$, $\varphi_i^{0|4}$ and $\varphi_{ij}^{0|5}$ are small. Formally, however, the distinguishing feature of the one-dimensional, rotationally symmetric geometry is that the expansion coefficients $\varphi_i^{0|4}$ and $\varphi_{ij}^{0|5}$ vanish identically and that the expansion coefficient $\varphi^{0|3}$ may be arbitrary.

$$\partial_t \epsilon + \partial_i q^i = 0, \tag{2.1a}$$

$$\partial_t q^i + \partial_j \left(\frac{c^2}{3} \delta^{ij} \epsilon + M^{ij} \right) = -\frac{1}{\tau_r} q^i, \tag{2.1b}$$

$$\partial_t M^{ij} + \partial_k \left(\frac{2c^2}{5} \delta^{k\langle i} q^{j\rangle} + M^{ijk} \right) = -\frac{1}{\tau_r} M^{ij} - \frac{1}{\tilde{\tau}_n} \left(M^{ij} - \frac{3c}{2c\epsilon + \sqrt{4c^2\epsilon^2 - 3|\mathbf{q}|^2}} q^{\langle i} q^{j\rangle} \right), \tag{2.1c}$$

where c is the constant Debye speed, $\tau_r = \tau_r(\epsilon)$ is the effective relaxation time for resistive processes and $\tilde{\tau}_n = \tilde{\tau}_n(\epsilon)$ is the effective relaxation time for normal processes. As usual, angle brackets denote the symmetric tracefree part, e.g.,

$$q^{\langle i} q^{j\rangle} := q^i q^j - \tfrac{1}{3} |\mathbf{q}| \delta^{ij}. \tag{2.2}$$

The moment flux M^{ijk}, which is a rank 3 symmetric tracefree tensor, has the form

$$M^{ijk} = -\frac{S}{4\epsilon^2 E} q^{\langle i} q^j q^{k\rangle} - \frac{1}{A} \left[\frac{3B}{\epsilon} q^{\langle i} M^{jk\rangle} - \frac{1}{c^2\epsilon^3 D} \left(2L\, q^l M_l{}^{\langle i} q^j q^{k\rangle} - \frac{Q}{c^2\epsilon^2 E} M_{lm} q^l q^m q^{\langle i} q^j q^{k\rangle} \right) \right], \tag{2.3}$$

where the coefficients (A, B, D, E, L, Q, S) depend on $(\epsilon, \mathbf{q})$ according to the relations

$$A := \frac{1}{u^2} \left[\frac{(1-u)^2}{2\sqrt{u}} \ln \left(\frac{1+\sqrt{u}}{1-\sqrt{u}} \right) + \frac{1}{3}(5u - 3) \right], \qquad A > 0, \tag{2.4a}$$

$$u := \frac{3\big(2c\epsilon - \sqrt{4c^2\epsilon^2 - 3|\mathbf{q}|^2}\big)}{2c\epsilon + \sqrt{4c^2\epsilon^2 - 3|\mathbf{q}|^2}}, \qquad B := \frac{3+u}{4u} \left(\frac{8}{3} - 5A \right), \tag{2.4b}$$

$$D := \frac{2}{3} - A > 0, \qquad E := 3(3-u)A - 4 > 0, \tag{2.4c}$$

$$L := \left(\frac{3+u}{4} \right)^3 \frac{1}{u^2} \left[\frac{15}{8} (3+u) A^2 - 13A + \frac{16}{3} \right], \tag{2.4d}$$

$$Q := \left(\frac{3+u}{4} \right)^5 \frac{1}{u^3} \left[\frac{45}{4} (1-u^2) A^3 + 3(19u - 27) A^2 - 4(7u - 15) A - \frac{32}{3} \right], \tag{2.4e}$$

$$S := \left(\frac{3+u}{4} \right)^2 \frac{1}{u} \left[\frac{45}{2} (1-u)(3-u) A^2 + 6(39 - 23u) A - 16(9 - 5u) \right]. \tag{2.4f}$$

The derivation of (2.1) and (2.3) is based on an expansion about a quasi-equilibrium Planck distribution F. (Alternatively, we call F the anisotropic Planck function.) Since F depends on the heat flux in a non-perturbative manner, there are effectively no unphysical limitations on the value of $|\mathbf{q}|$, i.e., one can handle problems with large components of the heat flux. This is a definite improvement over previous approaches [20–22] which only make allowances for small deviations in the heat flux from zero. Also, due to the explicit presence of two relaxation times $(\tau_r, \tilde{\tau}_n)$ in equation (2.1c), the 9-moment system is expected to be a useful tool in dealing with both normal and resistive processes. Moreover, assuming a separation of two time scales $(\tilde{\tau}_n \ll \tau_r)$, one can treat phenomena at frequencies comparable to the inverse of the normal time. However, since the infinite set of the expansion coefficients $(\varphi^{n|m})$ was truncated and the moment flux M^{ijk} was approximated by a linear function of M^{ij}, a limitation of the 9-moment system is that it is incapable of representing the effects of large departures from local quasi-equilibrium.

We are now ready to present a study of the one-dimensional, rotationally symmetric reduction of equations (2.1*a*)–(2.1*c*). Prior to that, however, we require some preliminary definitions. Let v^i be defined by

$$v^i := \frac{3+u}{4c\epsilon} q^i = \frac{3}{2c\epsilon + \sqrt{4c^2\epsilon^2 - 3|\mathbf{q}|^2}} q^i. \tag{2.5}$$

From (2.4*b*) and (2.5) it follows that

$$u = \delta_{ij} v^i v^j = (v^1)^2 + (v^2)^2 + (v^3)^2. \tag{2.6}$$

Upon introducing the useful quantity

$$N^{ij} := M^{ij} - \frac{4c^2\epsilon}{3+u} v^{\langle i} v^{j\rangle} = M^{ij} - \frac{3c}{2c\epsilon + \sqrt{4c^2\epsilon^2 - 3|\mathbf{q}|^2}} q^{\langle i} q^{j\rangle} \tag{2.7}$$

and setting $M_{ijk} := M^{ijk}$, equations (2.3)–(2.7) enable us to obtain the identity

$$M_{ijk} v^i v^j v^k = cu \left[\frac{6c^2\epsilon}{5(3+u)} G u^2 - \frac{I}{E} N_{ij} v^i v^j \right], \tag{2.8}$$

where

$$G := \frac{1}{2u}\left[\frac{8}{3} - 5(1-u)A\right], \qquad I := \frac{3}{u}[3(3u-5)A + 4(2-u)]. \tag{2.9}$$

For further details, including a derivation of (2.8), the reader is referred to [5, appendix B].

As noted already in the introduction, the equations of 9-moment phonon hydrodynamics are not easy to analyse, and the difficulty in analysing these equations lies in the complicated structure of the moment flux and collisional terms [5]. In order to examine the hyperbolicity question, it is very tempting to try deriving evolution equations with the same basic features as the original ones, but simpler to study. For our purposes, a considerable but nontrivial simplification can be achieved by assuming the one-dimensional, rotationally symmetric geometry. In this geometry, all gas-state variables are functions of time and a single spatial coordinate $x := x^1$. The heat flux in the x-direction $q(t, x)$ may vary ($q := q^1$); the heat fluxes q^2, q^3 in the orthogonal directions are set equal to zero ($q^2 = q^3 = 0$). A rank 2 symmetric tracefree tensor M^{ij} specializes to $M^{11} = -2M^{22} = -2M^{33}$. For essentially one-dimensional problems, i.e., problems with rotational symmetry about the x-axis, there are no off diagonal components of M^{ij}. As regards the moment flux M^{ijk} which is a rank 3 symmetric tracefree tensor [5], we can use the component M^{111} as a basis for the representation of this moment flux. Here, it is convenient to introduce the change of variables

$$m := \tfrac{3}{2} M^{11}, \qquad M := \tfrac{3}{2} M^{111}. \tag{2.10}$$

These new variables will be of interest to us subsequently.

With these observations in mind, equations (2.1*a*)–(2.1*c*) reduce to a system of three equations for (ϵ, q, m):

$$\partial_t \epsilon + \partial_x q = 0, \tag{2.11a}$$

$$\partial_t q + \partial_x \left(\frac{c^2}{3}\epsilon + \frac{2}{3} m\right) = -\frac{1}{\tau_r} q, \tag{2.11b}$$

$$\partial_t m + \partial_x \left(\frac{2c^2}{5} q + M\right) = -\frac{1}{\tau_r} m - \frac{c^2\epsilon}{\tilde{\tau}_n} N, \tag{2.11c}$$

where

$$N := \frac{m}{c^2\epsilon} - \frac{4u}{3+u}. \tag{2.12}$$

Here, in view of $q^2 = q^3 = 0$ and $q := q^1$, we have

$$u = |v|^2 = (v)^2, \tag{2.13}$$

where

$$v := v^1 = \frac{3}{2c\epsilon + \sqrt{4c^2\epsilon^2 - 3|q|^2}}\, q \qquad (v^2 = v^3 = 0). \tag{2.14}$$

There remains, however, an additional quantity M corresponding to the moment flux M^{ijk}. In order for system (2.11) to be a determined system of equations for (ϵ, q, m), it is necessary that M is uniquely characterized by specifying (ϵ, q, m). Implicitly, much of the analysis of the one-dimensional, rotationally symmetric reduction of the equations of 9-moment phonon hydrodynamics has been presented in [5, appendix B]. There, the concrete nonlinear expression for $M_{ijk}v^i v^j v^k$, equation (2.8), was derived. With the aid of this expression, we easily find that M can be related to (ϵ, q, m) by means of the following formula[6]:

$$M = \frac{c^3\epsilon v}{3+u}\left[\frac{9}{5}\, G u - \frac{I}{E}\,(3+u)\, N\right]. \tag{2.15}$$

In the one-dimensional geometry, the quantities $M_{ijk}v^i v^j v^k$ and $N_{ij}v^i v^j$ may be rewritten as $M_{ijk}v^i v^j v^k = 2Muv/3$ and $N_{ij}v^i v^j = 2c^2\epsilon Nu/3$. Then, since $u = |v|^2 = (v)^2$, equation (2.8) simplifies to yield (2.15). System (2.11), in conjunction with equations (2.12)–(2.15), (2.4*c*), (2.9) and (2.4*a*), forms a closed set of equations from which the evolution of (ϵ, q, m) can in principle be determined.

Starting from the kinetic-theory definitions of ϵ and q, we verify that q is subject to the condition $|q| < c\epsilon$ (see [5, footnote 5]). Inserting this condition into (2.14) yields the additional constraints of the form

$$|v| < 1, \qquad u < 1. \tag{2.16}$$

For a fixed value of ϵ, the variation of v with q, in the range $-c\epsilon < q < c\epsilon$, is monotonic. This is proved by noting that, from (2.14), $\partial v/\partial q > 0$. The quantities v and u vanish at $q = 0$, and the variations of A and E with u are such that

$$0 < A < \tfrac{2}{3}, \qquad E > 0. \tag{2.17}$$

(Regarding the proof of (2.17), see [5, section 3.4].) Moreover, in the limit $u \to 0_+$, we obtain [5]

$$A = \tfrac{8}{15}, \qquad E = \tfrac{4}{5}, \qquad G = \tfrac{8}{7}, \qquad I = -\tfrac{36}{35}. \tag{2.18}$$

In the light of these observations, we may say that the quantities appearing in (2.15) are regular functions of ϵ and q.

2.2. Evolution equations in first-order quasi-linear form

On account of the fact that system (2.11) is written in divergence form, it will (by analogy with physical laws) be said to be expressed in generalized conservation form. It is in strict conservation form when $\tau_r^{-1} = \tilde{\tau}_n^{-1} = 0$, for non-zero terms with τ_r^{-1} and $\tilde{\tau}_n^{-1}$ correspond to the occurrence of either generalized sources or productions. Our purpose here is to show

[6] Formally, in the limit $u \to 1_-$, we obtain $G = 4/3$ and $I/E = -3$.

that equations (2.11a)–(2.11c) can be written as a first-order, quasi-linear system. However, in order to derive this system, we need to introduce the following auxiliary formulae:

$$\frac{\partial v}{\partial \epsilon} = -\frac{v(3+u)}{\epsilon(3-u)}, \qquad \frac{\partial v}{\partial q} = \frac{(3+u)^2}{4c\epsilon(3-u)}, \tag{2.19a}$$

$$\frac{\partial u}{\partial \epsilon} = -\frac{2u(3+u)}{\epsilon(3-u)}, \qquad \frac{\partial u}{\partial q} = \frac{v(3+u)^2}{2c\epsilon(3-u)}, \tag{2.19b}$$

$$\frac{\mathrm{d}A}{\mathrm{d}u} = \frac{8+3(u-5)A}{6u(1-u)}, \qquad u \to 0_+ \Rightarrow \frac{\mathrm{d}A}{\mathrm{d}u} = \frac{8}{105}. \tag{2.19c}$$

Direct differentiation of $(2c^2/5)\,q + M$ with respect to x then gives

$$\partial_x\left(\frac{2c^2}{5}\,q + M\right) = \frac{3}{2}\,c^3 a\,\partial_x \epsilon + \frac{3}{2}\,c^2 b\,\partial_x q + cd\,\partial_x m, \tag{2.20}$$

where

$$a := -\frac{v[9(5+u)A^2 - 84A + 32]}{3(3-u)A - 4} + \frac{v(3+u)}{3-u}\,YN, \tag{2.21a}$$

$$b := \frac{81(1+u)A^2 - 12(21+2u)A + 112}{3[3(3-u)A-4]} - \frac{(3+u)^2}{4(3-u)}\,YN, \tag{2.21b}$$

$$d := -\frac{3[3(3u-5)A + 4(2-u)]}{v[3(3-u)A-4]} \tag{2.21c}$$

and $3(3-u)A - 4 > 0$ (see (2.4c)). In (2.21a) and (2.21b), the quantity Y is defined by

$$Y := \frac{9[9(u^2-2u+5)A^2 - 24(1+u)A + 16u]}{u[3(3-u)A-4]^2}. \tag{2.22}$$

For any fixed value of $\epsilon > 0$, as q moves towards 0, one can prove that

$$Y = -\frac{9}{7}, \qquad a = 0, \qquad b = \frac{4}{15} + \frac{27}{28}\,\frac{m}{c^2\epsilon}, \qquad d = 0. \tag{2.23}$$

In addition to this, the investigation of (2.22) yields the following result (see the appendix): Y as a function of u satisfies the inequality $Y < 0$. One of the main applications of this inequality is to the discussion of the hyperbolicity question (see sections 3.4 and 3.5).

Before explicitly showing that equations (2.11a)–(2.11c) can be cast into a first-order quasi-linear form, it is both natural and convenient to define (w^i), (P_i) and (B_{ij}) by

$$(w^i) := (\epsilon, q, m), \tag{2.24a}$$

$$(P_i) := \left(0, -\frac{1}{\tau_r}\,q, -\frac{1}{\tau_r}\,m - \frac{c^2\epsilon}{\tilde{\tau}_n}\,N\right), \tag{2.24b}$$

$$(B_{ij}) := \begin{pmatrix} 0 & 1 & 0 \\ c^2/3 & 0 & 2/3 \\ 3c^3 a/2 & 3c^2 b/2 & cd \end{pmatrix}, \tag{2.24c}$$

where, of course, $i, j = 1, 2, 3$. Let δ_{ij} be the Kronecker delta. Then the introduction of the above notation enables system (2.11) to be written as

$$\delta_{ij}\,\partial_t w^j + B_{ij}\,\partial_x w^j = P_i \qquad (i = 1, 2, 3). \tag{2.25}$$

(Here and throughout this paper, we adopt the summation convention whereby a repeated index implies summation over all values of that index. Setting $\delta^{ij} := \delta_{ij}$, the indices such as i and j will be raised and lowered with δ^{ij} and δ_{ij}, respectively.)

Since B_{ij} and P_i are functions of (w^i) only, system (2.25) is a first-order, quasi-linear system as it stands. This system will be the starting point for our further investigations.

3. Symmetric hyperbolicity

3.1. Left symmetric symmetrizers

The most general symmetric 3×3 matrix has the form

$$(S_{ij}) := \begin{pmatrix} S_{11} & S_{12} & \mu \\ S_{12} & S_{22} & \nu \\ \mu & \nu & \rho \end{pmatrix}. \tag{3.1}$$

Here we have used the notation

$$\mu := S_{13}, \qquad \nu := S_{23}, \qquad \rho := S_{33}. \tag{3.2}$$

Assume the entries of (S_{ij}) are functions of (w^i). Also, suppose (S_{ij}) is non-singular: $\det(S_{ij}) \neq 0$. Premultiplying the equation

$$\delta_{kj}\,\partial_t w^j + B_{kj}\,\partial_x w^j = P_k \tag{3.3}$$

by

$$S_i{}^k := S_{il}\,\delta^{lk} \tag{3.4}$$

and employing the summation convention, we find that

$$S_{ij}\,\partial_t w^j + C_{ij}\,\partial_x w^j = Q_i \qquad (i = 1, 2, 3), \tag{3.5}$$

where

$$C_{ij} := S_i{}^k B_{kj}, \qquad Q_i := S_i{}^k P_k. \tag{3.6}$$

This set of evolution equations is equivalent to system (2.25).

For essentially obvious reasons, system (3.5) will be symmetric if the entries of (C_{ij}) satisfy the conditions

$$C_{ij} = C_{ji} \qquad (i, j = 1, 2, 3). \tag{3.7}$$

Assuming that the introduction of (S_{ij}) gives us precisely these conditions, we call (S_{ij}) a left symmetric symmetrizer. If (S_{ij}) is a left symmetric symmetrizer, then the entries of (S_{ij}) are characterized by

$$S_{11} = \tfrac{1}{2}\,c^2\left[\mu + c\nu(3a - d) - 3\left(\mu - \tfrac{1}{2}\,c^2\rho\right)b\right], \tag{3.8a}$$

$$S_{12} = \tfrac{1}{2}\,c\left(c\nu + \tfrac{9}{2}\,c^2\rho a - 3\mu d\right), \tag{3.8b}$$

$$S_{22} = \tfrac{3}{2}\left(\mu + \tfrac{3}{2}\,c^2\rho b - c\nu d\right). \tag{3.8c}$$

Ignoring for the moment the condition $\det(S_{ij}) \neq 0$, we are justified in saying that μ, ν and ρ are arbitrary functions of (w^i). Consequently, equations (3.1) and (3.8) define not a single symmetrizer, but a whole family of symmetrizers.

Since one can change the symmetrizer by changing the functions (μ, ν, ρ), it is desirable to take advantage of this freedom to make the symmetrizer positive definite for significant ranges of physical conditions. Then applying (S_{ij}) to equations (2.25) yields a symmetric hyperbolic system of the form (3.5) (with non-zero sources), so the existence and uniqueness results of [13] hold. These and similar issues are discussed in the text below.

3.2. A positive-definite symmetrizer

An interesting symmetrizer is obtained by setting

$$\mu = \frac{4u}{3c^2}, \qquad \nu = -\frac{2v}{c^3}, \qquad \rho = \frac{3-u}{3c^4}. \tag{3.9}$$

It then follows immediately from (2.21) and (3.8) that the remainder of specification of the symmetrizer is straightforward:

$$S_{11} = \frac{1}{12}\left[27(1+5u)A - 4(16u+3) - \frac{9}{4}(1+3u)Z\right], \tag{3.10a}$$

$$S_{12} = -\frac{v}{4c}[9(5+u)A - 20 - 3Z], \tag{3.10b}$$

$$S_{22} = \frac{3}{4c^2}\left[9(1+u)A - 4 - \frac{1}{4}(3+u)Z\right]. \tag{3.10c}$$

As regards the meaning of Z, this quantity is defined in terms of N by

$$Z := (3+u)YN. \tag{3.11}$$

For the phonon gas described by (ϵ, q, m), a quasi-equilibrium state is the one in which $N = 0$; equivalently, we have (see (2.12))

$$m = \frac{4c^2\epsilon u}{3+u}. \tag{3.12}$$

In quasi-equilibrium, it may thus be concluded that the quantity Z vanishes.

In order to prove that the symmetrizer characterized by (3.9) and (3.10) is positive definite, we first consider the case of quasi-equilibrium. If $N = 0$, the matrix (S_{ij}) has an inverse, (R^{ij}):

$$R^{ik}S_{kj} = \delta^i{}_j, \qquad S_{ik}R^{kj} = \delta_i{}^j. \tag{3.13}$$

Clearly, this inverse is symmetric, in the sense that

$$R^{ij} = R^{ji} \qquad (i, j = 1, 2, 3). \tag{3.14}$$

Using the notation

$$\bar{R}^{ij} := \frac{3E}{(1-u)^2} R^{ij}, \tag{3.15}$$

we get

$$\bar{R}^{11} = 6\int_{-1}^{1} \frac{1}{(1-v\sigma)^5}\,\mathrm{d}\sigma = \frac{12(1+u)}{(1-u)^4}, \tag{3.16a}$$

$$\bar{R}^{12} = 6c\int_{-1}^{1} \frac{\sigma}{(1-v\sigma)^5}\,\mathrm{d}\sigma = \frac{4c(5+u)v}{(1-u)^4}, \tag{3.16b}$$

$$\bar{R}^{13} = 3c^2\int_{-1}^{1} \frac{3\sigma^2-1}{(1-v\sigma)^5}\,\mathrm{d}\sigma = \frac{24c^2u}{(1-u)^4}, \tag{3.16c}$$

$$\bar{R}^{22} = 6c^2\int_{-1}^{1} \frac{\sigma^2}{(1-v\sigma)^5}\,\mathrm{d}\sigma = \frac{4c^2(1+5u)}{(1-u)^4}, \tag{3.16d}$$

$$\bar{R}^{23} = 3c^3 \int_{-1}^{1} \frac{\sigma(3\sigma^2-1)}{(1-v\sigma)^5}\,\mathrm{d}\sigma = \frac{8c^3(1+2u)v}{(1-u)^4}, \tag{3.16e}$$

$$\bar{R}^{33} = \frac{3c^4}{2} \int_{-1}^{1} \frac{(3\sigma^2-1)^2}{(1-v\sigma)^5}\,\mathrm{d}\sigma = \frac{3c^4[9(1-u)^2A+4(3u-1)]}{(1-u)^4}. \tag{3.16f}$$

Now, let $(\alpha_1, \alpha_2, \alpha_3)$ be the set consisting of three real numbers satisfying the condition $\delta^{ij}\alpha_i\alpha_j \neq 0$. For each $(\alpha_1, \alpha_2, \alpha_3)$ with the above property, an elementary calculation leads directly to the result

$$R^{ij}\alpha_i\alpha_j = \frac{(1-u)^2}{2E} \int_{-1}^{1} \frac{\mathcal{B}^2}{(1-v\sigma)^5}\,\mathrm{d}\sigma, \tag{3.17}$$

in which $\mathcal{B}$ is defined by

$$\mathcal{B} := 2\alpha_1 + 2c\sigma\alpha_2 + c^2(3\sigma^2-1)\alpha_3. \tag{3.18}$$

Since $|v| = \sqrt{u} < 1$ and $E > 0$, this result implies that the matrix (R^{ij}) is positive definite. The rest of the argument is easy. Given that (R_{ij}) is an inverse of (S_{ij}), it follows from (3.13) that the matrix (S_{ij}) is also positive definite. Because of this last statement, system (3.5) is verified to be (algebraically) symmetric hyperbolic for all quasi-equilibrium states.

Passing now to the general case ($N \neq 0$), we need one concept from linear algebra. Namely, the matrix (S_{ij}) is positive definite if and only if the determinant of each leading principal minor is positive [24]. For (S_{ij}), there are three determinants defined by

$$\mathcal{D}_1 := S_{11}, \tag{3.19a}$$

$$\mathcal{D}_2 := S_{11}S_{22} - (S_{12})^2, \tag{3.19b}$$

$$\mathcal{D}_3 := \det(S_{ij}). \tag{3.19c}$$

In a quasi-equilibrium state ($N = 0$), the matrix (S_{ij}) is positive definite and we have the inequalities

$$\mathcal{D}_i > 0 \qquad (i = 1, 2, 3). \tag{3.20}$$

Thus, since $(\mathcal{D}_i)$ are continuous functions of (w^i), there exists a neighbourhood of a quasi-equilibrium state such that the determinants $(\mathcal{D}_i)$ are positive. This conclusion holds for every quasi-equilibrium state. In other words, we have demonstrated that system (3.5) is certainly symmetric hyperbolic in the neighbourhood of quasi-equilibrium, i.e., in the range where relation (2.15) is valid.

Of course, we expect that system (3.5) cannot be symmetric hyperbolic irrespective of how far from quasi-equilibrium the state is, and in order to calculate the region of symmetric hyperbolicity in parameter space (the space defined by either ϵ, q, m or ϵ, q, $\varphi^{0|3}$), we must propose a more detailed investigation of the conditions $\mathcal{D}_i > 0$. A full treatment of these conditions is given in section 3.4. Subsequently, section 3.5 proposes the passage from (ϵ, q, m) to $(\epsilon, q, \varphi^{0|3})$, which is a diffeomorphic change of variables, and also derives the symmetric hyperbolic system for $(\epsilon, q, \varphi^{0|3})$. In section 3.3, we turn our attention to an approximate expression for the Boltzmann entropy and then show that this expression is of direct physical relevance to the specification of (μ, ν, ρ) via (3.9).

3.3. Relation to the approximate kinetic entropy

Neglecting in [5, equation (3.51)] the terms that involve the expansion coefficients other than $\left(\varphi^{0|3}, \varphi_i^{0|4}, \varphi_{ij}^{0|5}\right)$, an approximate expression for the Boltzmann entropy is given by

$$s = s_F\left[1 - \frac{E}{8(3-u)}(\varphi^{0|3})^2 - \frac{3}{4}uD\,\delta^{ij}\varphi_i^{0|4}\varphi_j^{0|4} - \frac{3}{8}A\,\delta^{ij}\delta^{kl}\varphi_{ik}^{0|5}\varphi_{jl}^{0|5}\right], \tag{3.21}$$

where

$$s_F := 4k_B\epsilon\Delta\left(\frac{1-u}{3+u}\right), \qquad \Delta := \frac{\chi}{\epsilon^{1/4}}\frac{(3+u)^{1/4}}{(1-u)^{3/4}}, \qquad \chi := \left(\frac{\pi^2}{30c^3\hbar^3}\right)^{1/4}. \tag{3.22}$$

As usual, we denote by k_B the Boltzmann constant and by $\hbar$ the Planck constant divided by 2π. For the one-dimensional, rotationally symmetric geometry, as a consequence of formulae [5, (4.6*a*)–(4.6*c*)], we easily verify that the expansion coefficients $\left(\varphi_i^{0|4}, \varphi_{ij}^{0|5}\right)$ vanish identically,

$$\varphi_i^{0|4} = 0, \qquad \varphi_{ij}^{0|5} = 0, \tag{3.23}$$

and that the quantities $\varphi^{0|3}$ and N are related by

$$\varphi^{0|3} = -\frac{(9-u^2)N}{3(1-u)E}. \tag{3.24}$$

Then (3.21) may be written in the form

$$s = k_B\epsilon\Delta\left[\frac{4(1-u)}{3+u} - \frac{9-u^2}{18(1-u)}\frac{N^2}{E}\right]. \tag{3.25}$$

Here, as with the formula for s_F, Δ is defined by (3.22).

Evaluating the right-hand side of

$$H_{ij} := \left(\frac{\partial^2 s}{\partial w^i \partial w^j}\right)_{N=0} \tag{3.26}$$

requires[7] a twice differentiable evaluation of the scalar function s in terms of $(w^i) := (\epsilon, q, m)$. Of course, such an evaluation of $s = s(w^i)$ is going to be quite complicated, but this is the price to be paid for the structural simplicity that emerges below. Indeed, a tedious calculation using (2.12), (2.4*c*), (3.22) and (2.19*b*) shows that for every quasi-equilibrium state one has

$$(H_{ij}) = -\frac{k_B\Delta}{3\epsilon E}\left(\frac{3+u}{1-u}\right)(S_{ij}) \qquad (i, j = 1, 2, 3), \tag{3.27}$$

where (S_{ij}) has exactly the same meaning as in section (3.2), i.e., (S_{ij}) is derived from (3.9) and (3.10) by substituting $Z = 0$ into (3.10*a*)–(3.10*c*). Hence we find that (H_{ij}) is a negative-definite matrix. In problems where $N = 0$, this matrix can be used to define (S_{ij}) without appeal to the algebraic construction of sections 3.1 and 3.2.

Now, with the aid of (3.2), we obtain for the basic matrix elements (H_{13}, H_{23}, H_{33}) the following formulae:

$$H_{13} = -\frac{k_B\Delta}{3\epsilon E}\left(\frac{3+u}{1-u}\right)\mu, \tag{3.28a}$$

$$H_{23} = -\frac{k_B\Delta}{3\epsilon E}\left(\frac{3+u}{1-u}\right)\nu, \tag{3.28b}$$

$$H_{33} = -\frac{k_B\Delta}{3\epsilon E}\left(\frac{3+u}{1-u}\right)\rho. \tag{3.28c}$$

These formulae clearly show the connection between (H_{13}, H_{23}, H_{33}) and (μ, ν, ρ). This is essential because, as explicitly seen in sections 3.1 and 3.2, the functions (μ, ν, ρ) given by (3.9) are relevant to our algorithm for defining a positive-definite symmetrizer (S_{ij}) for system (2.25).

[7] We set $m = 4c^2\epsilon u/(3+u)$ in $\partial^2 s/\partial w^i \partial w^j$. Consequently, the entries of (H_{ij}) are independent of m and (H_{ij}) is not exactly the same thing as the Hessian matrix of $s = s(w^i)$.

The above discussion has laid out the physical framework of the algebraic construction of positive-definite symmetrizers. However, an important issue concerning how to find the symmetrizer (if it does exist) which is a positive-definite (negative-definite) Hessian matrix of some scalar function of (w^i) still remains to be addressed. Note that the positive-definite symmetrizer (S_{ij}) defined in section 3.2 is not a symmetrizer of this type. Also, the Hessian matrix of $s = s(w^i)$ is not a left symmetric symmetrizer for system (2.25).

3.4. The region of symmetric hyperbolicity

Let (S_{ij}) be the symmetrizer characterized by (3.1), (3.9) and (3.10). As before, suppose that $(\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3)$ are the determinants of leading principal minors. Here, we recall that these determinants are given by (3.19). Define $(\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_2)$ by

$$\mathcal{E}_1 := 12\mathcal{D}_1, \qquad \mathcal{E}_2 := 16c^2\mathcal{D}_2, \qquad \mathcal{E}_3 := \frac{256c^6}{9(1-u)^2(3-u)}\,\mathcal{D}_3. \tag{3.29}$$

In agreement with the discussion of section 3.2, one obtains

$$\mathcal{E}_1 > 0, \qquad \mathcal{E}_2 > 0, \qquad \mathcal{E}_3 > 0 \tag{3.30}$$

as necessary and sufficient conditions for a symmetrizer, (S_{ij}), to be positive definite.

The quantities $(\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3)$ can be written explicitly. Evidently, using (3.29), (3.19*a*) and (3.10*a*), we verify that $\mathcal{E}_1$ is simply

$$\mathcal{E}_1 = 27(1+5u)A - 4(16u+3) - \tfrac{9}{4}(1+3u)Z. \tag{3.31}$$

Setting

$$Z_0 := \frac{4[27(1+5u)A - 4(16u+3)]}{9(1+3u)} \qquad (Z_0 > 0), \tag{3.32}$$

we find that $\mathcal{E}_1 > 0$ if and only if the following condition is satisfied:

$$Z < Z_0. \tag{3.33}$$

It is easy to confirm that $Z_0 > 0$ because $\mathcal{E}_1 = 27(1+5u)A - 4(16u+3) > 0$ if $Z = 0$. Given (3.29) as well as (3.19*c*) and (3.10), the explicit form of $\mathcal{E}_3$, on the other hand, appears at first sight rather complicated, being

$$\mathcal{E}_3 = Z^2 - \frac{8E}{3-u}Z + \frac{16E^2}{3(3-u)}\,. \tag{3.34}$$

However, a little algebra shows that it simplifies to

$$\mathcal{E}_3 = (Z - Z_1)(Z - Z_2), \tag{3.35}$$

where

$$Z_1 := \frac{4E}{3(3-u)}(3 - \sqrt{3u}) \qquad (Z_1 > 0), \tag{3.36a}$$

$$Z_2 := \frac{4E}{3(3-u)}(3 + \sqrt{3u}) \qquad (Z_2 \geqslant Z_1). \tag{3.36b}$$

Then we have $\mathcal{E}_3 > 0$ if $Z < Z_1$. The inequality $\mathcal{E}_3 > 0$ also arises when $Z > Z_2$, but the reader will verify that in this case it is not possible to guarantee that $Z < Z_0$.

It only remains to calculate the quantity $\mathcal{E}_2$ and interpret the inequality $\mathcal{E}_2 > 0$. With the aid of (3.29), (3.19*b*) and (3.10), the required expression for $\mathcal{E}_2$ is readily seen to be

$$\begin{aligned}\mathcal{E}_2 = {} & \tfrac{27}{16}(1-u)^2 Z^2 - \tfrac{1}{2}[81(1-u)^2 A - 4(8u^2 - 21u + 9)]Z \\ & + 3[9(1-u)^2 A + 4(3u-1)]E.\end{aligned} \tag{3.37}$$

Since $\mathcal{E}_2$ is a function of u and Z (see (2.4a) and (2.4c)),

$$\mathcal{E}_2 = \mathcal{E}_2(u, Z), \tag{3.38}$$

this function can be evaluated at $Z = Z_1$ and $Z = Z_2$; in detail, we have

$$\mathcal{E}_2(u, Z_1) = 16E\sqrt{\frac{u^3}{3}}\left[\frac{(\sqrt{3}-\sqrt{u})(\sqrt{3}+2\sqrt{u})}{3-u}\right]^2, \tag{3.39a}$$

$$\mathcal{E}_2(u, Z_2) = -16E\sqrt{\frac{u^3}{3}}\left[\frac{(\sqrt{3}+\sqrt{u})(\sqrt{3}-2\sqrt{u})}{3-u}\right]^2. \tag{3.39b}$$

Hence we conclude that $\mathcal{E}_2(u, Z_1)$ and $\mathcal{E}_2(u, Z_2)$ obey the conditions

$$\mathcal{E}_2(u, Z_1) \geqslant 0, \qquad \mathcal{E}_2(u, Z_2) \leqslant 0. \tag{3.40}$$

Evaluating $\mathcal{E}_2(u, Z)$ at $Z = 0$ and $Z = Z_0$ gives[8]

$$\mathcal{E}_2(u, 0) = 3[9(1-u)^2 A + 4(3u-1)]E, \tag{3.41a}$$

$$\mathcal{E}_2(u, Z_0) = -u\,[9(5+u)A - 20 - 3Z_0]^2. \tag{3.41b}$$

Knowing that $E > 0$ and $\mathcal{E}_2(u, 0) > 0$, from these formulae it follows at once that

$$9(1-u)^2 A + 4(3u-1) > 0, \qquad \mathcal{E}_2(u, Z_0) \leqslant 0. \tag{3.42}$$

The essential thing to note here is that the quantity $\mathcal{E}_2(u, Z_0)$ is not positive.

Clearly, the equation $\mathcal{E}_2(u, Z) = 0$ is a quadratic equation for Z, yielding two real values (Z_3, Z_4) which may be regarded as functions of u:

$$\mathcal{E}_2(u, Z) = \tfrac{27}{16}(1-u)^2(Z - Z_3)(Z - Z_4). \tag{3.43}$$

Without any loss of generality, we assume that

$$Z_3 \leqslant Z_4. \tag{3.44}$$

Now, by what has been said above, it is not difficult to verify that the values of Z_0, Z_1, Z_2, Z_3 and Z_4 are restricted by the inequalities

$$Z_3 \leqslant Z_0 \leqslant Z_4, \tag{3.45a}$$

$$0 < Z_1 \leqslant Z_3 \leqslant Z_2 \leqslant Z_4. \tag{3.45b}$$

Given these inequalities, our basic result can be stated very neatly. If $Z < Z_1$, then the matrix (S_{ij}) is positive definite. Using (2.12) and (3.36a) and remembering that $Z := (3+u)YN$ and $Y < 0$ (see equation (2.22) and the appendix), one finds that the condition $Z < Z_1$ is equivalent to the inequality

$$m > \frac{4c^2\epsilon u}{3+u} - \frac{4c^2\epsilon(3-\sqrt{3u})E}{3(9-u^2)|Y|}. \tag{3.46}$$

Since the quantities (u, E, Y) depend only on u and u is an explicit function of (ϵ, q) (see (2.4c), (2.4a), (2.22), (2.13) and (2.14)), this last inequality plus an obvious bound of the form

$$|q| < c\epsilon \tag{3.47}$$

[8] For $Z = Z_0$, we have $S_{11} = 0$, $\mathcal{D}_2 = -(S_{12})^2$ and $\mathcal{E}_2 = -(4cS_{12})^2$. This explains why $\mathcal{E}_2(u, Z_0)$ can be written in the form (3.41b).

define precisely the region of symmetric hyperbolicity in the (ϵ, q, m)-space, i.e., the region for which system (3.5) is symmetric hyperbolic. It is then easy to explain why equations (2.25) form a hyperbolic system when $Z < Z_1$.

We denote the aforementioned region by $\mathcal{W}$. Note that $\mathcal{W}$ is a convex subset of $\mathbb{R}^3$. Another important observation is that every quasi-equilibrium state is contained in $\mathcal{W}$:

$$\frac{4c^2\epsilon u}{3+u} > \frac{4c^2\epsilon u}{3+u} - \frac{4c^2\epsilon(3-\sqrt{3u})E}{3(9-u^2)|Y|}\,. \tag{3.48}$$

We also mention the following. In the case when

$$N := \frac{m}{c^2\epsilon} - \frac{4u}{3+u} \geqslant 0, \tag{3.49}$$

there are effectively no mathematical limitations on the magnitude of N as condition (3.46) is then satisfied automatically. The inequality $Z < Z_1$ and the equivalent condition (3.46) give only an estimate of the domain of symmetric hyperbolicity. Needless to say, this estimate does not enable us to conclude that the original system (2.25) is necessarily non-hyperbolic in the region where $Z \geqslant Z_1$. Whether system (2.25) is non-hyperbolic for $Z \geqslant Z_1$ is an open problem that remains to be seen.

3.5. Transformation of gas-state variables

Since the solution for m of equations (2.12) and (3.24) has the form

$$m = \frac{c^2\epsilon}{3+u}\left[4u - \frac{3(1-u)}{3-u}\,E\varphi^{0|3}\right], \tag{3.50}$$

the passage from (ϵ, q, m) to $(\epsilon, q, \varphi^{0|3})$ is a diffeomorphic change of gas-state variables. Remembering that $(w^i) := (\epsilon, q, m)$ and abbreviating $(\epsilon, q, \varphi^{0|3})$ as (ω^i), we can use (3.50), (2.4*c*) and (2.19) to derive the following formulae:

$$\partial_t w^i = D^i{}_j\,\partial_t\omega^j, \qquad \partial_x w^i = D^i{}_j\,\partial_x\omega^j, \tag{3.51}$$

where

$$\left(D^i{}_j\right) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \vartheta & \xi & \gamma \end{pmatrix}, \qquad \gamma := -\frac{3c^2\epsilon(1-u)}{9-u^2}\,E, \tag{3.52a}$$

$$\vartheta := -\frac{4c^2u}{3-u} - \frac{18c^2}{(3-u)^3}[3(3-u)^2A - 2(7-4u+u^2)]\varphi^{0|3}, \tag{3.52b}$$

$$\xi := \frac{6cv}{3-u} - \frac{3c}{4v(3-u)^3}[3(u^2-10u-15)(3-u)^2A + 8(27-5u^2+2u^3)]\varphi^{0|3}. \tag{3.52c}$$

The coefficients (ϑ, ξ, γ) tend to the limits

$$\vartheta = -\frac{4c^2}{15}\,\varphi^{0|3}, \qquad \xi = 0, \qquad \gamma = -\frac{4c^2\epsilon}{15} \tag{3.53}$$

as q approaches 0. A little algebra, aided by $\gamma < 0$, yields $\det\left(D^i{}_j\right) = \gamma < 0$. Hence we conclude that the matrix $\left(D^i{}_j\right)$ is non-singular if $|q| < c\epsilon$.

Using (3.5) and (3.51), and the fact that the entries of (S_{ij}), (C_{ij}) and (Q_i) can be expressed as functions of (ω^i), the system of equations for (ω^i) reads

$$S_{kl}D^l{}_j\,\partial_t\omega^j + C_{kl}D^l{}_j\,\partial_x\omega^j = Q_k \qquad (k = 1, 2, 3). \tag{3.54}$$

Suppose, given

$$Z := (3+u)YN = -\frac{3(1-u)}{3-u}\, EY\varphi^{0|3}, \tag{3.55}$$

that (S_{ij}) is defined as in section 3.2. Then the matrix (C_{ij}) is symmetric and the matrix (S_{ij}) is symmetric and positive definite. Now, if we premultiply (3.54) by $D^k{}_i$ and employ the summation convention, we arrive at

$$E_{ij}\,\partial_t\omega^j + F_{ij}\,\partial_x\omega^j = U_i \qquad (i = 1, 2, 3), \tag{3.56}$$

where

$$U_i := D^k{}_i Q_k, \qquad E_{ij} := D^k{}_i S_{kl} D^l{}_j, \qquad F_{ij} := D^k{}_i C_{kl} D^l{}_j. \tag{3.57}$$

Elementary inspection shows that

$$E_{ij} = E_{ji}, \qquad F_{ij} = F_{ji}. \tag{3.58}$$

Also, we have

$$E_{ij}\alpha^i\alpha^j = S_{kl}\left(D^k{}_i\alpha^i\right)\left(D^l{}_j\alpha^j\right) > 0 \tag{3.59}$$

if $\delta_{ij}\alpha^i\alpha^j \neq 0$. Because of (3.58) and (3.59), system (3.56) is symmetric hyperbolic.

Using (3.47) as well as (3.46) and (3.50), the region for which system (3.56) is symmetric hyperbolic is characterized by

$$|q| < c\epsilon, \qquad \varphi^{0|3} < \Xi := \frac{4(3-\sqrt{3u})}{9(1-u)|Y|}. \tag{3.60}$$

If $\varphi^{0|3} < 0$, the second condition in (3.60) is satisfied automatically. For any fixed value of ϵ, as q approaches 0, one can show that $\varphi^{0|3} < \Xi = 28/27$. In the limit $|q| \to (c\epsilon)_-$, one obtains $\varphi^{0|3} < \Xi = \infty$. Consequently, the differential equations for $(\epsilon, q, \varphi^{0|3})$ form a symmetric hyperbolic system even beyond the limits of their original derivation ($|\varphi^{0|3}| \ll 1$), and indeed this type of observation is one of the most unexpected features of the one-dimensional reduction of the equations of 9-moment phonon hydrodynamics.

4. Characteristic speeds

Hyperbolicity of equations (2.25) can be investigated by examining the eigenstructure of (B_{ij}) (see (2.24*c*)). These equations are hyperbolic if the eigenvalues of (B_{ij}) are all real and their corresponding eigenvectors are distinct. As is well-recognized (see, e.g., [25]), any real $n \times n$ matrix (B_{ij}) has n real eigenvalues and n linearly independent eigenvectors if and only if it has a real positive-definite left symmetrizer. Therefore, the explicit construction of such a symmetrizer in section 3 suffices to guarantee that equations (2.25) are hyperbolic in $\mathcal{W}$. For the waves of weak discontinuity, an eigenvalue describes the characteristic wavespeed of propagation, the corresponding right eigenvector gives the hydrodynamic or quasi-hydrodynamic quantities transported by the wave and the left eigenvector prescribes the strength of the disturbance. In an analysis of system (2.25), the following characteristic equation is obtained:

$$\det(B_{ij} - c\lambda\delta_{ij}) = 0, \tag{4.1}$$

where λ is an eigenvalue divided by c. An equivalent statement to (4.1) is, of course, a cubic equation of the form

$$P(\lambda) := -\lambda^3 + \mathrm{d}\lambda^2 + \left(\tfrac{1}{3} + b\right)\lambda + a - \tfrac{1}{3}\mathrm{d} = 0. \tag{4.2}$$

Before proceeding further, let us define Φ by

$$\Phi := -\left[\tfrac{1}{3}\left(\tfrac{1}{3}+b\right)+\left(\tfrac{1}{3}\mathrm{d}\right)^2\right]^3+\left[\tfrac{1}{27}\mathrm{d}^3+\tfrac{1}{6}\mathrm{d}\left(\tfrac{1}{3}+b\right)+\tfrac{1}{2}\left(a-\tfrac{1}{3}\mathrm{d}\right)\right]^2. \tag{4.3}$$

The quantity Φ has an especial interest. In the region $\mathcal{W}$ where equations (2.25) transform into a symmetric hyperbolic system, it satisfies the inequality $\Phi \leqslant 0$, one consequence of which is that the zeros $(\lambda_1, \lambda_2, \lambda_3)$ of the cubic polynomial $P(\lambda)$ are real.

In principle, it is possible to derive analytic expressions for $(\lambda_1, \lambda_2, \lambda_3)$ in terms of (ϵ, q, N), but these expressions are not very instructive. To get a sense of how the roots of $P(\lambda) = 0$ behave for small values of N (i.e., in a neighbourhood of quasi-equilibrium states), we employ a polynomial approximation to

$$\lambda_i = \lambda_i(\epsilon, q, N) \qquad (i = 1, 2, 3). \tag{4.4}$$

To the lowest nontrivial order in N, this approximation implies that

$$\lambda_i = \eta_i + \kappa_i N + O(N^2), \tag{4.5}$$

where η_i and κ_i are functions of (ϵ, q). Denoting by (a_o, b_o) those parts of (a, b) which do not depend on N, we can write equations (2.21a) and (2.21b) in the form

$$a = a_o + \frac{v(3+u)}{3-u}\,YN, \qquad b = b_o - \frac{(3+u)^2}{4(3-u)}\,YN. \tag{4.6}$$

Since (η_1, η_2, η_3) are the zeros of

$$P_o(\eta) := -\eta^3 + \mathrm{d}\,\eta^2 + \left(\tfrac{1}{3}+b_o\right)\eta + a_o - \tfrac{1}{3}\mathrm{d} = 0, \tag{4.7}$$

we conclude that to a first approximation

$$P(\lambda_i) = \frac{1}{4}\left\{\frac{4}{3}\left(1+3b_o+6\mathrm{d}\eta_i-9\eta_i^2\right)\kappa_i + \frac{3+u}{3-u}\,[4v-(3+u)\eta_i]\,Y\right\}N + O(N^2). \tag{4.8}$$

The above formula for $P(\lambda_i)$ will be consistent with the equation $P(\lambda_i) = 0$ if the coefficient in front of N vanishes. Then the quantity κ_i can be evaluated; it turns out to be

$$\kappa_i = -\frac{3(3+u)[4v-(3+u)\eta_i]Y}{4(3-u)\left(1+3b_o+6\,\mathrm{d}\eta_i-9\eta_i^2\right)}\,. \tag{4.9}$$

Here η_i is a solution for η of the equation $P_o(\eta) = 0$.

Comparatively simple results are obtained only when the absolute value of $z := q/(c\epsilon)$ is much smaller than 1. For if $|z| \ll 1$, one can expand all functions of z as polynomials. When this expansion technique is applied to $P_o(\eta_i)$, the equation $P_o(\eta_i) = 0$ gives

$$\eta_1 = \frac{1}{5}\sqrt{15} + \frac{3}{14}z - \frac{9}{490}\sqrt{15}z^2 + \frac{447}{5488}z^3 + O(z^4), \tag{4.10a}$$

$$\eta_2 = -\frac{1}{5}\sqrt{15} + \frac{3}{14}z + \frac{9}{490}\sqrt{15}z^2 + \frac{447}{5488}z^3 + O(z^4), \tag{4.10b}$$

$$\eta_3 = \frac{15}{28}z + \frac{3417}{21\,952}z^3 + O(z^4), \tag{4.10c}$$

and then

$$\kappa_1 = \frac{9}{56}\sqrt{15} - \frac{585}{1568}z + O(z^2), \tag{4.11a}$$

$$\kappa_2 = -\frac{9}{56}\sqrt{15} - \frac{585}{1568}z + O(z^2), \tag{4.11b}$$

$$\kappa_3 = \frac{585}{784}z + O(z^2). \tag{4.11c}$$

We now see clearly the following. The characteristic speeds depend on the non-equilibrium state characterized by the values of z and N. In equilibrium ($z = N = 0$), equation (4.2) degenerates into a trivial equation

$$-\lambda\left(\lambda^2 - \frac{3}{5}\right) = 0, \tag{4.12}$$

so that the characteristic speeds are 0 and $\pm\sqrt{15}c/5$.

5. Discussion and final remarks

Our method can be used to present a systematic derivation of a whole hierarchy of closed systems of moment equations. The system of equations for the energy density and the heat flux is the basic, non-perturbative member of this hierarchy of closures [4]. In [5], we have investigated in detail the next member, the 9-moment closure model. The one-dimensional, rotationally symmetric reduction of this model appears to be an interesting one as it reveals a nontrivial system of three evolution equations which, for a well-defined region of parameter space, is a symmetrizable hyperbolic system.

As is well-recognized, the method of maximum entropy leads automatically to a hierarchy of moment closure systems, each of which possesses an entropy and is symmetric hyperbolic [16–19]. However, the difficulty is that explicit closed-form expressions for the coefficients of the maximum-entropy distribution function in terms of the moments prove to be impossible to obtain. Moreover, cases are known where the equilibrium states are located on the boundary of the domain of definition of the maximum-entropy system [18, 23]. These complications do not arise in the Grad-type approach [20, 21]. Therefore, despite the disadvantage concerning non-positivity of the approximate distribution function, the perturbative expansion technique seems to be favourable.

The present study was inspired by the work of Groth *et al* [26, 27]. Within the framework of classical kinetic theories, these authors considered using an expansion about an ellipsoidal distribution function (EDF) as opposed to the expansion based on perturbations to a Maxwellian. The use of the EDF accounts for arbitrary pressure anisotropies and hyperbolicity of the moment closure system is possible for a well-defined region of parameter space. In fact, their closure technique may be regarded as a practical compromise between the non-perturbative method of Levermore [17] and standard perturbative methods [20, 21].

In [5], the equations of 9-moment phonon hydrodynamics were derived by expanding about an anisotropic Planck function and including the flux of the heat flux in the expansion. This moment closure prescription is very much analogous to that based on an expansion about an EDF. Note that the basic advantage of using the anisotropic Planck function is that the heat flux is incorporated into the model in a non-perturbative manner, thereby allowing virtually arbitrarily large values for the components of this heat flux. If the normal processes dominate the phonon distribution ($\tilde{\tau}_n \ll \tau_r$), this is a definite improvement over previous approaches which only make allowances for small deviations in the heat flux from zero; see especially [22].

We now mention the following. In extended hydrodynamic approaches, there is a difficult question that needs to be answered when addressing how gas flows should be computed: what are the boundary conditions to use with the extended systems of evolution equations? If one considers the shock problem or the problem of waves of weak discontinuity propagating into a region in equilibrium, the issue of determining a suitable set of boundary conditions is circumvented, in the sense that the boundary conditions are associated with the equilibrium distribution function. In other cases, however, one needs a criterion in order to prescribe the so-called non-controllable boundary data. For stationary processes, an interesting attempt to overcome this problem was proposed by Struchtrup and Weiss [28], who postulated that the non-controllable data are the ones for which the L^{∞} norm of the entropy production is minimal. Their postulate became known as the minimax principle. Although this principle is capable of determining the missing boundary conditions, it leads to temperature fields that run counter to physical intuition and, more importantly, have never been observed.

A second effort in the analysis of a boundary-value problem was the study, by Liu *et al* [29], of one-dimensional heat conduction in the 14-moment classical system. They recalled the iterative procedure for the moment equations invented by Maxwell, and employed an iteration of a similar spirit. Iterated values for a physically non-controllable boundary value were calculated by requiring that, in each iterative step, the approximate solution is close to the exact one. A third approach was proposed by Brini and Ruggeri [30] which consists in utilizing the concept of a critical derivative. Restricting attention to system (2.25), there is still uncertainty as to which of the three, if any, approaches presented above is the best for our model. As a matter of fact, this diversity of proposals emphasizes the need of further discussion and additional search for a comprehensive and satisfactory formulation of the boundary-value problem.

Even for well-posed boundary conditions, complexity of the moment flux and collisional terms makes system (2.25) hardly solvable by purely analytical means. Thus, it is clear that the appropriate numerical methods must be proposed. The hyperbolic structure of the moment equations evident in our analysis lends itself to solution techniques that take advantage of the wave-like nature of the transport phenomena. Such techniques require the development of good approximate Riemann solvers. In the dissertation by Brown [31], a Roe-type approximate Riemann solver for the 35-moment classical model was presented. We expect that many of the ideas discussed in [31], when appropriately modified, will be useful in the study of numerical schemes for system (2.25).

Also, a natural question to be addressed is whether system (2.25) implies a new balance law, interpreted as the equation of balance of entropy. Consequently, the new density associated with this equation is the entropy density, which is a function of the original gas-state variables. In the method of maximum entropy, one knows fully well that the additional balance law can be derived rather easily by rearranging the system of evolution equations into symmetric conservative form (see, e.g., [32]). For system (2.25), the question is open and any result on the existence of the formal (i.e., mathematical) entropy density would be interesting. Of course, this formal density, if it exists at all, must be different from the approximate kinetic-theory density constructed in section 3.3. Nevertheless, we believe that such a result could facilitate an analysis of the waves of strong discontinuity. As regards the effective strategy for finding an expression for the mathematical entropy, this goal can be achieved by requiring that system (2.25) admits the Hessian matrix as a symmetrizer. The corresponding formal entropies are then determined by direct integration of the admissible Hessian symmetrizers [12].

Acknowledgment

This work was supported by the State Committee for Scientific Research (KBN) through the grant no 5 T07A 046 24.

Appendix. Proof of the inequality $Y < 0$

Using (2.4*a*), (2.4*c*) and (2.22), the quantity Y can be written in the form

$$Y = \frac{9(1-u)^4(\Theta_1 - \Theta_2)(\Theta_1 + \Theta_2)}{u^5(u^2 - 2u + 5)E^2}, \tag{A.1}$$

where

$$\Theta_1 := 3(u^2 - 2u + 5)R - 15 + u, \tag{A.2a}$$

$$R := \frac{1}{2\sqrt{u}} \ln\left(\frac{1+\sqrt{u}}{1-\sqrt{u}}\right), \qquad \Theta_2 := \frac{4u^2}{\sqrt{1-u}}. \tag{A.2b}$$

With the aid of

$$R = \sum_{n=0}^{\infty} \frac{1}{2n+1} u^n, \tag{A.3}$$

we obtain for Θ_1

$$\Theta_1 = 12u^2 \sum_{n=0}^{\infty} \frac{4n^2 + 8n + 5}{(2n+1)(2n+3)(2n+5)} u^n. \tag{A.4}$$

For Θ_2, we have

$$\Theta_2 = 4u^2 \sum_{n=0}^{\infty} \frac{(2n-1)!!}{(2n)!!} u^n, \tag{A.5}$$

where

$$(2n-1)!! := 1 \cdot 3 \cdot 5 \cdots (2n-1), \tag{A.6a}$$

$$(2n)!! := 2 \cdot 4 \cdot 6 \cdots (2n), \tag{A.6b}$$

$$(-1)!! := 1, \qquad (0)!! := 1. \tag{A.6c}$$

Consequently, it is clear that

$$\frac{1}{u^3}(\Theta_1 - \Theta_2) < 0, \qquad \frac{1}{u^2}(\Theta_1 + \Theta_2) > 0. \tag{A.7}$$

Since $u^2 - 2u + 5 > 0$ and $E > 0$, this immediately proves the inequality $Y < 0$.

References

[1] Peierls R E 1955 *Quantum Theory of Solids* (London: Oxford University Press)
[2] Beck H 1975 Second sound and related thermal conduction phenomena *Dynamical Properties of Solids* vol 2 ed G K Horton and A A Maradudin (Amsterdam: North-Holland) p 205
[3] Gurevich V L 1980 *Kinetika Fononnykh Sistem* (Moscow: Nauka)
[4] Larecki W 1992 *Nuovo Cimento* D **14** 141
[5] Banach Z and Larecki W 2004 Nine-moment phonon hydrodynamics based on the modified Grad-type approach: formulation *J. Phys. A: Math. Gen.* **37** 9805
[6] Banach Z and Piekarski S 1989 *J. Math. Phys.* **30** 1826
[7] Callaway J 1991 *Quantum Theory of the Solid State* (Boston, MA: Academic)
[8] Jeffrey A 1976 *Quasilinear Hyperbolic Systems and Waves* (London: Pitman)
[9] Majda A 1984 *Compressible Fluid Flow and Systems of Conservation Laws in Several Space Variables* (Berlin: Springer)
[10] Smoller J 1983 *Shock Waves and Reaction-Diffusion Equations* (New York: Springer)

[11] Larecki W 1997 *Arch. Mech.* **49** 865

[12] Larecki W 1998 Matrix analysis approach to determination of entropies for hyperbolic systems of conservation equations *Trends in Continuum Physics* ed B T Maruszewski, W Muschik and A Radowicz (Singapore: World Scientific) p 202

[13] Taylor M E 1996 *Partial Differential Equations. III. Nonlinear Equations* (New York: Springer)

[14] Friedrichs K O and Lax P D 1971 *Proc. Natl Acad. Sci. USA* **68** 1686

[15] Courant R and Hilbert D 1962 *Methods of Mathematical Physics* vol II (New York: Interscience)

[16] Dreyer W 1987 *J. Phys. A: Math. Gen.* **20** 6505

[17] Levermore C D 1996 *J. Stat. Phys.* **83** 1021

[18] Banach Z and Larecki W 2002 *Rev. Math. Phys.* **14** 469

[19] Banach Z 2003 *Class. Quantum Grav.* **20** 1443

[20] Grad H 1949 *Commun. Pure Appl. Math.* **2** 331

[21] Grad H 1958 Principles of the kinetic theory of gases *Handbuch der Physik* vol 12 ed S Flügge (Berlin: Springer) p 205

[22] Banach Z and Piekarski S 1989 *J. Math. Phys.* **30** 1816

[23] Junk M 1998 *J. Stat. Phys.* **93** 1143

[24] Hoffman K and Kunze R 1961 *Linear Algebra* (Englewood Cliffs, NJ: Prentice-Hall)

[25] Jameson A, Kreindler E and Lancaster P 1992 *Linear Algebr. Appl.* **160** 189

[26] Groth C P T, Gombosi T I, Roe P L and Brown S L 1994 Gaussian-based moment-method closures for the solution of the Boltzmann equation *5th Int. Conf. on Hyperbolic Problems (Stony Brook, NJ, June 1994)*

[27] Groth C P T, Roe P L, Gombosi T I and Brown S L 1995 On the nonstationary wave structure of a 35-moment closure for rarefied gas dynamics *AIAA 26th Fluid Conf. (San Diego, CA, June 1995)*

[28] Struchtrup H and Weiss W 1998 *Phys. Rev. Lett.* **80** 5048

[29] Liu I S, Rincon M A and Müller I 2002 *Continuum Mech. Thermodyn.* **14** 483

[30] Brini F and Ruggeri T 2002 *Continuum Mech. Thermodyn.* **14** 165

[31] Brown S L 1996 Approximate Riemann solvers for moment models of dilute gases *Dissertation* University of Michigan

[32] Müller I and Ruggeri T 1998 *Rational Extended Thermodynamics* (New York: Springer)